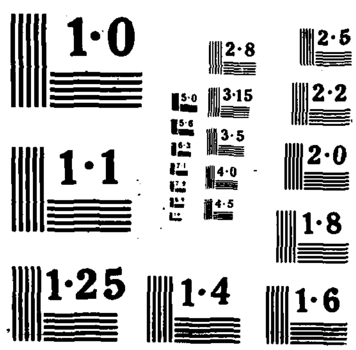
AD-A172 582 ANALYSIS OF A SINGULARLY-PERTURBED LINEAR TWO-POINT BOUNDARY-VALUE PROBLEM(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER W E FERGUSON JUL 86 1/1

UNCLASSIFIED MRC-TSR-2941 DAAG29-80-C-0041 F/G 12/1 NL

END
DATE
FILMED
11-86


1·0
1·1
1·25
1·4
1·6
1·8
2·0
2·2
2·5
2·8
3·15
3·5
4·0
4·5
5·0
5·6
6·3

MRC Technical Summary Report #2941

ANALYSIS OF A SINGULARLY-PERTURBED LINEAR TWO-POINT BOUNDARY-VALUE PROBLEM

Warren E. Ferguson, Jr.

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

July 1986

(Received February 26, 1986)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

U. S. Department of Energy
Washington, DC 20545

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

ANALYSIS OF A SINGULARLY-PERTURBED LINEAR TWO-POINT BOUNDARY-VALUE PROBLEM

Warren E. Ferguson, Jr.

Technical Summary Report #2941

July 1986

ABSTRACT

Consider as $\varepsilon \to 0^+$, the solution

$$y(x) = \{1 - \exp(-x/\varepsilon)\} / \{1 - \exp(-1/\varepsilon)\}$$

of the following singularly-perturbed linear two-point boundary-value problem

$$\varepsilon\, y''(x) + y'(x) = 0, \quad y(0) = 0, \quad y(1) = 1 \quad .$$

When this problem is cast as a system of first-order differential equations, using the substitution $y_1 = y$ and $y_2 = y'$, one is led to consider the problem

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \frac{d}{dx} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad .$$

As $\varepsilon \to 0^+$ the solution of this first order system has the property that $y_2(0)$ blows up like ε^{-1} while $y_1(x)$ remains uniformly bounded on $[0,1]$. This paper presents a simple constructive method of solving such singularly-perturbed linear two-point boundary-value problems; a method which has been generalized to analyze the solution of such problems by finite different schemes based on Euler's method.

AMS (MOS) Subject Classifications: 34B27, 34E15, 65L10

Key Words: Ordinary Differential Equations, Boundary-Value Problem, Singularly-Perturbed, Green's Function, Constructive Methods, A Priori Bounds, Difference Schemes.

Work Unit Numbers 1 & 3 - Applied Analysis; Numerical Analysis and Scientific Computing

Mathematics Department, Southern Methodist University, Dallas, TX 75275.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the U. S. Department of Energy under Contract No. DE-AL02-82ER12046A000.

SIGNIFICANCE AND EXPLANATION

This paper presents a simple constructive proof of the existence of solutions to a class of singularly-perturbed linear two-point boundary-value problems. Such problems arise when detailed models of physical phenomena involve effects that occur on markedly different temporal or spatial scales, the (singularly) small parameter measuring the disparity between the scales. This simple constructive proof, through slight modification, also allows one to prove the existence of solutions of a class of difference schemes used to approximate the solution of the continuous problem.

ANALYSIS OF A SINGULARLY-PERTURBED LINEAR TWO-POINT BOUNDARY-VALUE PROBLEM

Warren E. Ferguson, Jr.

1. Introduction

When a high-order singularly-perturbed linear two-point boundary-value problem is cast as a first-order system of differential equations, it is usually the case that the solution of this first order system does not remain uniformly bounded as the small parameter tends to zero. This leads us to consider, as $\varepsilon \to 0^+$, a differential equation on $[0,1]$ of the form

$$\begin{bmatrix} \varepsilon I & & \\ & I & \\ & & \varepsilon I \end{bmatrix} \frac{d}{dx} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} - \begin{bmatrix} A_{11}(x) & A_{12}(x) & \varepsilon A_{13}(x) \\ A_{21}(x) & A_{22}(x) & A_{23}(x) \\ \varepsilon A_{31}(x) & A_{32}(x) & A_{33}(x) \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}$$

subject to appropriate boundary conditions. Here it is assumed that the real parts of the eigenvalues of $A_{11}(x)$ are negative while those of $A_{33}(x)$ are positive. In this paper reasonable general conditions are placed on the boundary condition which ensure that the solution y of this differential equation satisfies the following:

<u>Requirement 1.1</u>: For all sufficiently small positive ε

$$\lim_{\varepsilon \to 0^+} \begin{bmatrix} \varepsilon y_1(0) \\ y_2(0) \\ \varepsilon y_3(1) \end{bmatrix}$$

exists and is finite.

Problems of this type have been considered by other authors [6,11]. What distinguishes the analysis presented in this paper is the fact that a straightforward analog can also be used to analyze the stability and consistency properties of difference

Mathematics Department, Southern Methodist University, Dallas, TX 75275.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the U. S. Department of Energy under Contract No. DE-AL02-82ER12046A000.

schemes used to solve this problem.

In section 2 we precisely define the class of singularly-perturbed linear two-point boundary-value problems to be studied. In sections 3 and 4 a simple constructive technique is presented which allows one to characterize, in a reasonable general manner, when such singularly-perturbed boundary-value problems admit a solution satisfying Requirement 1.1. In section 5 applications of the results presented in sections 3 and 4 are described.

2. Preliminaries

Consider the following two boundary-value problems $BV(\nu)$ for $\nu = G, S$ which differ only in the boundary conditions imposed on their solutions:

$$\mathcal{L}y = f \quad \text{and} \quad \mathcal{B}^{(\nu)}y = g^{(\nu)} \quad ,$$

where

$$(\mathcal{L}y)(x) \equiv \begin{bmatrix} \varepsilon I & & \\ & I & \\ & & \varepsilon I \end{bmatrix} \frac{d}{dx} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} - \begin{bmatrix} A_{11}(x) & A_{12}(x) & \varepsilon A_{13}(x) \\ A_{21}(x) & A_{22}(x) & A_{23}(x) \\ \varepsilon A_{31}(x) & A_{32}(x) & A_{33}(x) \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

and

$$\mathcal{B}^{(\nu)}y \equiv B_0^{(\nu)}y(0) + B_1^{(\nu)}y(1) \quad .$$

The unique solvability of the general problem $BV(G)$ will be related to the unique solvability of the special problem $BV(S)$ under the following assumptions.

<u>Assumptions 2.1</u>: For all sufficiently small positive ε:

0) $n = n_1 + n_2 + n_3$,

1) the components $y_i(x)$, $f_i(x)$, and $g_i^{(\nu)}$ of the n-vectors $y(x)$, $f(x)$, and $g^{(\nu)}$ respectively are n_i-vectors,

2) the order n_i by n_j matrices $A_{ij}(x) = A_{ij}(x;\varepsilon)$ depend smoothly on x and ε,

3) the vector $f(x) = f(x;\varepsilon)$ depends smoothly on x and ε,

4) the order n matrices $B_0^{(G)} = B_0^{(G)}(\varepsilon)$, $B_1^{(G)} = B_1^{(G)}\varepsilon$ and order n vectors $g^{(\nu)} = g^{(\nu)}(\varepsilon)$, depend smoothly on ε,

5) $B_0^{(S)} = \begin{bmatrix} \varepsilon I & & \\ & I & \\ & & 0 \end{bmatrix}$ and $B_1^{(S)} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & \varepsilon I \end{bmatrix}$ are order n matrices,

6) for some positive constant μ independent of ε the eigenvalues of $A_{11}(x)$ have real part less than $-\mu$ while the eigenvalues of $A_{22}(x)$ have real part greater than $+\mu$.

By smooth dependence on x and/or ε we mean that all possible derivatives with respect to x and/or ε exist and are continuous.

When one formally sets $\varepsilon = 0$ the differential equation $Ly = f$ reduces to the following familiar reduced system of algebraic-differential equations

$$y_1(x) = -A_{11}^{-1}(x)[A_{12}(x)y_2(x) + f_1(x)] \quad ,$$

$$\frac{d}{dx} y_2(x) = A_{22}^{*}(x)y_2(x) + f_2^{*}(x) \quad , \text{ and}$$

$$y_3(x) = -A_{33}^{-1}(x)[A_{32}(x)y_2(x) + f_3(x)]$$

where

$$A_{22}^{*}(x) \equiv A_{22}(x) - A_{21}(x)A_{11}^{-1}(x)A_{12}(x) - A_{23}(x)A_{33}^{-1}(x)A_{32}(x) \quad , \text{ and}$$

$$f_2^{*}(x) \equiv f_2(x) - A_{21}(x)A_{11}^{-1}(x)f_1(x) - A_{23}(x)A_{33}^{-1}(x)f_3(x) \quad .$$

When assumptions 2.1 hold, standard results in singular perturbation theory suggest that as $\varepsilon \to 0^+$ the solution of BV(S) approximately satisfies this reduced system of algebraic-differential equations in each fixed open subinterval of $[0,1]$.

The proof of the unique solvability of BV(S) begins by casting the BV(S) as an equivalent operator equation acting on an appropriate Banach space. To cast BV(S) into an equivalent operator equation we will use the fundamental solution matrices $Y_1(x,s)$, $Y_2(x,s)$, and $Y_3(x,s)$ defined as follows:

$$\varepsilon \frac{\partial}{\partial x} Y_1(x,s) = A_{11}(x)Y_1(x,s) \quad , \quad Y_1(s,s) = I \quad ,$$

$$\frac{\partial}{\partial x} Y_2(x,s) = A_{22}^{*}(x)Y_2(x,s) \quad , \quad Y_2(s,s) = I \quad , \text{ and}$$

$$\varepsilon \frac{\partial}{\partial x} Y_3(x,s) = A_{33}(x)Y_3(x,s) \quad , \quad Y_3(s,s) = I \quad .$$

Assumptions 2.1 and well-known results for initial-value problems guarantee the existence of these fundamental solution matrices.

The results presented in this paper depend primarily on three fundamental theorems. The first theorem relates the unique solvability of BV(G) to the unique solvability of BV(S).

Theorem 2.2: Let BV(S) be uniquely solvable. Define $Y^{(S)}$ to be the fundamental solution matrix of BV(S), that is

$$LY^{(S)} = 0 \quad \text{and} \quad B^{(S)}Y^{(S)} = I \quad .$$

Then BV(G) is uniquely solvable if and only if the matrix $B^{(G)}Y^{(S)}$ is nonsingular.

This theorem, whose proof can be found in [8,9], was used by Keller to show that a difference scheme is stable and consistent for a uniquely solvable boundary-value problem if and only if it is stable and consistent for the related initial-value problem. The proof of this theorem for difference equations rest on Kron's method of tearing [12]; the boundary conditions of the boundary-value problem are torn from the matrix describing the difference scheme and replaced by the boundary conditions of the related initial-value problem. This idea of replacing one set of boundary conditions by another underlies the proof that BV(G) is uniquely solvable.

The second theorem, whose proof can be found in [4], provides needed estimates on the size of the fundamental solution matrices Y_1 and Y_3.

Theorem 2.3: There exist positive constants K and λ, independent of ε, such that

$$|Y_1(x,s)| \leq K\, e^{-\lambda(x-s)/\varepsilon} \quad \text{for} \quad 0 \leq s \leq x \leq 1 \quad , \text{ and}$$

$$|Y_3(x,s)| \leq K\, e^{-\lambda(s-x)/\varepsilon} \quad \text{for} \quad 0 \leq x \leq s \leq 1$$

for all sufficiently small positive ε.

In this theorem the norm $|\cdot|$ used is any fixed n-vector norm, say the infinity vector norm. Note that the condition that ε be sufficiently small is important [1,2]; for unless ε is sufficiently small $|Y_1(x,s)|$ need not decay with increasing $(x-s)$ even though the eigenvalues of $A_{11}(x)$ have uniformly negative real part.

The third theorem, whose proof can be found in [7], states that operators in a Banach space sufficiently near the identity are invertible. The Banach space C_n used in this theorem consists of the space of n-vector valued functions of x, continuous on $[0,1]$, endowed with the norm $\|\cdot\|$ defined by:

$$\|f(x)\| \equiv \max_{[0,1]} |f(x)| \quad .$$

Theorem 2.4: Let B be an operator on the Banach space C_n which admits the bound $\|B\| < 1$. Then $I-B$ is invertible with

$$(I-B)^{-1} = \sum_{\ell=0}^{\infty} B^{\ell}, \quad \text{and} \quad \|(I-B)^{-1} - \sum_{\ell=0}^{N} B^{\ell}\| \leq \frac{\|B\|^{N+1}}{1 - \|B\|} .$$

This theorem will be used to obtain a generalized power series expansion of the solution of both BV(S) and BV(G).

3. Solution of the Special Problem

Let us now show that, for all sufficiently small positive ε, the speical problem BV(S) is uniquely solvable. As stated in section 2, the first step is to cast BV(S) as an equivalent operator equation acting on the Banach space C_n.

<u>Lemma 3.1</u>: $y(x)$ is a solution of BV(S) if and only if it is a solution of the operator equation

$$Ly = F$$

where

$$L \equiv \begin{bmatrix} I & -K_1A_{12} & -\varepsilon K_1A_{13} \\ -K_2A_{21} & I-K_2(A_{22}-A_{22}^*) & -K_2A_{23} \\ -\varepsilon K_3A_{31} & -K_3A_{32} & I \end{bmatrix}, \text{ and}$$

$$F \equiv \begin{bmatrix} \varepsilon^{-1}Y_1(\cdot,0)g_1^{(S)} + K_1f_1 \\ Y_2(\cdot,0)g_2^{(S)} + K_2f_2 \\ \varepsilon^{-1}Y_3(\cdot,1)g_3^{(S)} + K_3f_3 \end{bmatrix}$$

with

$$(K_1w)\ (x) \equiv \varepsilon^{-1} \int_0^x Y_1(x,s)w(s)\ ds \quad ,$$

$$(K_2w)\ (x) \equiv \int_0^x Y_2(x,s)w(s)\ ds \quad , \text{ and}$$

$$(K_3w)\ (x) \equiv \varepsilon^{-1} \int_1^x Y_3(x,s)w(s)\ ds \quad .$$

Proof: Consider the initial-value problem

$$\frac{d}{dx} w(x) - C(x)w(x) = h(x) \quad \text{and} \quad w(0) = \eta \quad ,$$

whose fundamental solution matrix $Y(x,s)$ satisfies the matrix initial-value problem

$$\frac{\partial}{\partial x} Y(x,s) - C(x)Y(x,s) = 0 \quad \text{and} \quad Y(s,s) = 1 \quad .$$

The variation of parameters formula states that the solution of this initial-value problem admits the representation

$$w(x) = Y(x,0)\eta + \int_0^x Y(x,s)h(s)\,ds \quad .$$

Suppose then that $y(x)$ is a solution of BV(S). Applying the variation of parameters formula to each component of the differential equation $\mathcal{L}y = f$, and the boundary-conditions specified by $\mathcal{B}^{(S)}y = g^{(S)}$, shows that $y(x)$ is also a solution of the operator equation $Ly = F$. Conversely, suppose that $y(x)$ is a solution of the operator equation $Ly = F$. Then $y(x)$ clearly satisfies the boundary conditions $\mathcal{B}^{(S)}y = g^{(S)}$ and differentiation of each component of the operator equation shows that $y(x)$ is also a solution of $\mathcal{L}y = f$.

□

To solve the integral equation $Ly = F$ note that it is possible to compute, for all sufficiently small positive ε, an approximate inverse of L.

<u>Lemma 3.2</u>: For all sufficiently small positive ε

$$J_0 L = I - \varepsilon M$$

where

$$J_0 = \begin{bmatrix} I & & \\ & 0 & \\ & & I \end{bmatrix} + \begin{bmatrix} K_1A_{12} \\ I \\ K_3A_{32} \end{bmatrix} [K_2A_{21} \;\; I \;\; K_2A_{23}] \quad ,$$

$$M = \begin{bmatrix} K_1A_{12}K_2A_{23}K_3A_{31} & K_1A_{12}N_2 & K_1(A_{12}K_2A_{21}K_1+I)A_{13} \\ K_2A_{23}K_3A_{31} & N_2 & K_2A_{21}K_1A_{13} \\ K_3(A_{32}K_2A_{23}K_3+I)A_{31} & K_3A_{32}N_2 & K_3A_{32}K_2A_{21}K_1A_{13} \end{bmatrix} \quad , \text{ and}$$

$$N_2 = \varepsilon^{-1}K_2[A_{22} + A_{21}K_1A_{12} + A_{23}K_3A_{32} - A_{22}^*]$$

are operators on $\mathcal{C}_n$ which are bounded independently of ε.

Proof: It is a simple matter to multiply J_0 by L and verify the product is indeed $I - \varepsilon M$. To show that J_0 and M are bounded independently of ε, for all sufficiently small positive ε, one uses Theorem 2.3. For example, to prove that N_2 is bounded first change the order of integration in the double integrals yielding $(N_2w)(x)$, substitute for the fundamental solution matrices Y_1 and Y_3 the expressions

$$Y_1(x,s) = \varepsilon A_{11}^{-1}(x) \frac{\partial}{\partial x} Y_1(x,s), \quad \text{and} \quad Y_3(x,s) = \varepsilon A_{33}^{-1}(x) \frac{\partial}{\partial x} Y_3(x,s) \quad ,$$

then perform an integration by parts and recall the definition of $A_{22}^{*}(x)$ before applying Theorem 2.3.

□

Using Theorem 2.4 we can now obtain, for all sufficiently small positive ε, a convergent generalized power series expansion for L^{-1}, the fundamental solution matrix $Y^{(S)}$, and the solution.

Theorem 3.4: For all sufficiently small positive ε, L^{-1} exists and admits the following generalized power series expansion:

$$L^{-1} = [\sum_{\ell=0}^{\infty} (\varepsilon M)^{\ell}] J_0 \quad .$$

Corollary 3.5: For all sufficiently small positive ε, $Y^{(S)}$ exists and admits the following generalized power series expansion:

$$Y^{(S)} = [\sum_{\ell=0}^{\infty} (\varepsilon M)^{\ell}] J_0 \begin{bmatrix} \varepsilon^{-1} Y_1(\cdot,0) & & \\ & Y_2(\cdot,0) & \\ & & \varepsilon^{-1} Y_3(\cdot,1) \end{bmatrix} \quad .$$

Corollary 3.6: For all sufficiently small positive ε, $BV(S)$ has a unique solution y which admits the following generalized power series expansion:

$$y = [\sum_{\ell=0}^{\infty} (\varepsilon M)^{\ell}] J_0 \begin{bmatrix} \varepsilon^{-1} Y_1(\cdot,0) g_1^{(S)} + K_1 f_1 \\ Y_2(\cdot,0) g_2^{(S)} + K_2 f_2 \\ \varepsilon^{-1} Y_3(\cdot,0) g_3^{(S)} + K_3 f_3 \end{bmatrix} \quad .$$

As shown by Theorem 2.4, when the above infinite sums are truncated after the term $\ell=N$ the resulting approximations to L^{-1}, $Y^{(S)}$, and y are $\mathcal{O}(\varepsilon^N)$ accurate as $\varepsilon \to 0^+$.

When the solution y of $Ly = f$ satisfies Requirement 1.1 the following Corollary shows that $y_2(x)$ remains bounded on $[0,1]$ as $\varepsilon \to 0^+$. A more detailed calculation, such as that provided by the method of matched asymptotic expansions [3], is needed to show that when the solution y of $Ly = f$ satisfies Requirement 1.1 both $y_1(x)$ and $y_3(x)$ remain bounded on each fixed open subinterval of $[0,1]$ as $\varepsilon \to 0^+$.

Corollary 3.7: For all sufficiently small positive ε, BV(S) has a unique solution y which admits the following a priori bounds:

$$\|y_1\| \leqq K \{ |y_1(0)| + |y_2(0)| + \varepsilon|y_3(1)| + \|f_1\| + \|f_2\|_1 + \|f_3\|\} \quad ,$$

$$\|y_2\| \leqq K \{\varepsilon|y_1(0)| + |y_2(0)| + \varepsilon|y_3(1)| + \|f_1\| + \|f_2\|_1 + \|f_3\|\} \quad , \text{ and}$$

$$\|y_3\| \leqq K \{\varepsilon|y_1(0)| + |y_2(0)| + |y_3(1)| + \|f_1\| + \|f_2\|_1 + \|f_3\|\} \quad .$$

Here the norm $\|f\|_1$ of a function $f(x)$ is defined as

$$\|f\|_1 \equiv \int_0^1 |f(x)| \, dx \quad .$$

4. Solution of the General Problem

By combining Theorem 2.2 and Corollary 3.5 one arrives at a explicit method for determining the unique solvability of BV(G). Recall that by Theorem 2.2, BV(G) is uniquely solvable if and only if $B^{(G)}Y^{(S)}$ is nonsingular. To arrive at reasonably general conditions under which the solution of BV(G) exists and satisfies Requirement 1.1 let us follow Harris [6] and introduce the following definition.

Definition 4.1: BV(G) is said to be regular if and only if it admits Assumptions 2.1 and the matrix

$$R = \lim_{\varepsilon \to 0^+} B^{(G)}Y^{(S)}$$

exists and is nonsingular.

Note that the smoothness requirement of Assumptions 2.1 allow us to compute R using the equivalent formula

$$R = \lim_{\varepsilon \to 0^+} B^{(G)}(I+\varepsilon M)J_0 \begin{bmatrix} \varepsilon^{-1}Y_1(\cdot,0) & & \\ & Y_2(\cdot,0) & \\ & & \varepsilon^{-1}Y_3(\cdot,1) \end{bmatrix} .$$

Theorem 4.2: If BV(G) is regular then for all sufficiently small positive ε it is uniquely solvable and its solution y satisfies Requirement 1.1.

Proof: By a continuity argument the nonsingularity of R and the assumed smoothness properties of BV(G) guarantee that $B^{(G)}Y^{(S)}$ is nonsingular for all sufficiently small positive ε. Therefore, by Theorem 2.2, BV(G) is uniquely solvable for all sufficiently small positive ε. To verify that the solution of BV(G) satisfies Requirement 1.1 let us observe that it admits the representation

$$y = \left[\sum_{\ell=0}^{\infty} (\varepsilon M)^{\ell}\right]J_0 \begin{bmatrix} \varepsilon^{-1}Y_1(0,\cdot)\{\varepsilon y_1(0)\} + K_1 f_1 \\ Y_2(0,\cdot)\{ y_2(0)\} + K_2 f_2 \\ \varepsilon^{-1}Y_3(\cdot,1)\{\varepsilon y_3(1)\} + K_3 f_3 \end{bmatrix} .$$

The smoothness requirements in Assumptions 2.1 and the regularity of BV(G) assure us that the limit

$$\lim_{\varepsilon\to 0^+} \begin{bmatrix} \varepsilon y_1(0) \\ y_2(0) \\ \varepsilon y_3(1) \end{bmatrix} = \lim_{\varepsilon\to 0^+} \left\{\left\{ B^{(G)} Y^{(S)} \right\}^{-1} g^{(G)} - B^{(G)} \left[\sum_{\ell=0}^{\infty} (\varepsilon M)^{\ell} \right] J_0 \begin{bmatrix} K_1 f_1 \\ K_2 f_2 \\ K_3 f_3 \end{bmatrix} \right\}$$

exists and is finite.

□

5. Conclusions

Similar techiques can be used to establish the stability and consistency properties of finite difference approximations to $BV(\nu)$ for $\nu = G, S$ based on Euler's method. This finite difference scheme [3,10], with step size h, uses the backward Euler scheme to approximate the differential equation for y_1, the box scheme of Keller (centered Euler) to approximate the differential equation for y_2, and the forward Euler scheme to approximate the differential equation for y_3. This scheme is $\mathcal{O}(h^2)$ accurate as $h \to 0^+$, uniformly in ε, in each fixed open subinterval of $[0,1]$.

The a priori estimates of Corollary 3.7 can also be used, as in [3], to show that the method of matched asymptotic expansions does indeed yield an asymptotically correct estimate of the solution of $BV(G)$ as $\varepsilon \to 0^+$. These a priori estimates can also be used to prove the existence of solutions of a class of nonlinear singularly perturbed boundary-value problems when the method of matched asymptotic expansions is coupled with the Newton-Kantorovich theorem.

More general boundary-value problems of the form

$$\begin{bmatrix} I & \\ & \varepsilon I \end{bmatrix} \frac{d}{dx} \begin{bmatrix} z_1(x) \\ z_2(x) \end{bmatrix} - \begin{bmatrix} C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x) \end{bmatrix} \begin{bmatrix} z_1(x) \\ z_2(x) \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} , \quad \mathcal{B}z = g$$

can also be analyzed by this technique under the restriction that the eigenvalues of $C_{22}(x)$ have real parts whose absolute value is bounded away from zero. For such problems a preliminary block diagonalization of $C_{22}(x)$ must be performed before the results of sections 3 and 4 can be applied. This block diagonalization of $C_{22}(x)$ involves the existence of a smooth nonsingular matrix $U(x)$ with the property that

$$C_{22}(x)U(x) = U(x) \begin{bmatrix} C_{22}^-(x) & \\ & C_{22}^+(x) \end{bmatrix}$$

where the real parts of the eigenvalues of $C_{22}^-(x)$ $\{C_{22}^+(x)\}$ are negative (positive). The existence of such a similarity transformation is established in [3,5].

These results summarize part of a thesis [3] written under the direction of Dr. H. B. Keller. I would like to thank Dr. Keller for suggesting this problem to me and for his supervision of my studies at the California Institute of Technology.

REFERENCES

[1] W. A. Coppel

"Stability and Asymptotic Behavior of Differential Equations," Heath, Boston, 1965.

[2] W. A. Coppel

"Dichotomies in Stability Theory," Springer-Verlag Lecture Notes in Mathematics, vol. 629, 1978.

[3] W. E. Ferguson, Jr.

"A Singularly Perturbed Linear Two-Point Boundary-Value Problem," Ph.D. Thesis, 1975, California Institute of Technology, and University Microfilms.

[4] L. Flatto and N. Levinson

"Periodic Solutions of Singularly Perturbed Systems," Journal of Rational Mechanics and Analysis, (1956) 4 (6), pp. 943-950.

[5] H. Gingold

"A Method of Global Blockdiagonalization for Matrix-Values Functions," SIAM J. Math. Anal., (1978) 9 (6), pp. 1076-1082.

[6] W. A. Harris, Jr.

"Singularly Perturbed Boundary-Value Problems Revisited," Springer-Verlag Lecture Notes in Mathematics, vol. 312, 1972.

[7] T. Kato

"Perturbation Theory for Linear Operators," Springer-Verlag, Die Grundlehren mathematischen Wissenschafter, Band 132, 1966.

[8] H. B. Keller

"Numerical Solution of Two-Point Boundary-Value Problems," SIAM CBMS-NSF Regional Conference Series in Applied Math., (1976) 24.

[9] H. B. Keller and A. B. White, Jr.

"Difference Methods for Boundary-Value Problems in Ordinary Differential Equations," SIAM J. Num. Anal. (1975) 12, pp. 791-802.

[10] H. O. Kreiss

"Difference Methods for Stiff Ordinary Differential Equations," SIAM J. Num. Anal. (1978) 15 (1).

[11] Robert E. O'Malley, Jr. and Joseph B. Keller

"Loss of Boundary Conditions in the Asymptotic Solution of Linear Ordinary Differential Equations, II Boundary Value Problems," Comm. Pure Ap. Math., 21 (1968), pp. 263-270.

[12] Ben Noble

"Applied Linear Algebra," Prentice-Hall, 1969.

WEF/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: #2941

2. GOVT ACCESSION NO.: AD-A172 582

3. RECIPIENT'S CATALOG NUMBER:

4. TITLE (and Subtitle): ANALYSIS OF A SINGULARLY-PERTURBED LINEAR TWO-POINT BOUNDARY-VALUE PROBLEM

5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER:

7. AUTHOR(s): Warren E. Ferguson, Jr.

8. CONTRACT OR GRANT NUMBER(s): DE-AL02-82ER12046A000, DAAG29-80-C-0041

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53705

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Work Unit Numbers 1 & 3 - Applied Analysis; Numerical Analysis and Scientific Computing

11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below

12. REPORT DATE: July 1986

13. NUMBER OF PAGES: 15

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report):
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):

18. SUPPLEMENTARY NOTES:

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

U. S. Department of Energy
Washington, DC 20545

19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
Ordinary Differential Equations, Boundary-Value Problem, Singularly-Perturbed, Green's Function, Constructive Methods, A Priori Bounds, Difference Schemes

20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

Consider as $\varepsilon \to 0^+$, the solution

$$y(x) = \{1 - \exp(-x/\varepsilon)\} / \{1 - \exp(-1/\varepsilon)\}$$

of the following singularly-perturbed linear two-point boundary-value problem

$$\varepsilon\, y''(x) + y'(x) = 0, \quad y(0) = 0, \quad y(1) = 1 \quad .$$

When this problem is cast as a system of first-order differential equations, using the substitution $y_1 = y$ and $y_2 = y'$, one is led to consider the

(continued)

DD FORM 1473, 1 JAN 73 — EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT (continued)

problem

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \frac{d}{dx} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

As $\varepsilon \to 0^+$ the solution of this first order system has the property that $y_2(0)$ blows up like ε^{-1} while $y_1(x)$ remains uniformly bounded on $[0,1]$. This paper presents a simple constructive method of solving such singularly-perturbed linear two-point boundary-value problems; a method which has been generalized to analyze the solution of such problems by finite different schemes based on Euler's method.

DATE

ILMED

8